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Optimization of Boundary Conditions for Maximum Fundamental Frequency of Vibrating Structures

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A sensitivity formula of eigenvalues with respect to the change of boundary conditions is derived using the material derivative concept based on a variational formulation. The change of boundary conditions is described with the introduction of the tangential component of the design velocity field used in shape design. Simply supported and partially welded plates are taken as numerical examples to check the accuracy of the sensitivity formula. The sensitivites of the distinct and multiple eigenvalues calculated by the formulas are compared with those calculated by finite differences. Optimal support locations are then determined by use of a gradient-based optimization method. It is shown that a crossing of eigenvalues can occur in the solution process.

Introduction

EARLIER works on structural optimization focused on the sizing design variables such as thickness and cross-sectional area. More recently, shape optimization has been a topic of intensive research and methods of shape design sensitivity analysis have been well-established in which the shape of a structure is taken as a design variable.[1,2]

In these problems, it is usually the case that support conditions are not considered design variables. However, the determination of support conditions is important because the structural performance can be improved dramatically by a simple adjustment of the support conditions. In the present study, a general formulation of a frequency response optimization in terms of boundary conditions is considered as a new category of problems. Although studies[3-5] on the sensitivity of frequency response with respect to the shape variation are available, there has been no concept of changing boundary conditions.

Wang and Nomura[6] have solved the problem of finding the optimal support locations of a free-free rectangular plate to maximize the fundamental natural frequency. The solution to the eigenvalue problem and the associated eigenvalue sensitivity analysis have been derived by use of the Rayleigh-Ritz formulation using symbolic algebra. Pitarresi and Kunz[7] and Kunz and Pitarresi[8] have proposed a simple and rapid method for approximating the optimal support locations of a vibrating plate by using a two-dimensional, nonlinear, least-square fit of the natural frequency against support location data. The advantage of their method is that it can be used with data from any analytical, computational, or experimental effort. But basically, this being an exhaustive search method, the estimated optimal results are dependent on the number of data and the order of fitting polynomials.

Recently, Keum and Kwak[9] have developed a general procedure for design sensitivity with changing boundary conditions by using the material derivative concept of continuum mechanics and applying a velocity field having tangential component. The formulation is then applied to calculate stress intensity factors.[10] The derivation of the sensitivity formula was based on a boundary integral equation formulation and the boundary element method was used for numerical implementation. This formulation is not efficient for eigenvalue analysis.

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In this paper, as shown in Ref. 9, the change of boundary conditions is described using a tangential variation of the subdomain containing the point separating segments of different boundary conditions. A simply supported beam and plates with various boundary conditions have been treated as numerical examples to demonstrate the accuracy of the sensitivity formula including multiple eigenvalues. Numerical results for optimal support locations of a simply supported plate are presented and the situation of eigenvalue crossing is investigated in detail.

Statement of the Problem

The problem treated in this research is the determination of the kinematically constrained boundary of a vibrating structure that maximizes the fundamental frequency. This boundary condition optimization problem can be stated as

$$\max_{\Gamma^u(b_i)} \left[\min_j \zeta_j \right] \tag{1a}$$

subject to

$$b_i \in I_i \qquad i = 1, \ldots, N \tag{1b}$$

$$a(u,\bar{u}) = \zeta d(u,\bar{u}) \qquad \text{for all } \bar{u} \in U_{\text{ad}} \tag{1c}$$

where ζ_j ($j = 1, \ldots, J$) denotes an eigenvalue that is possibly repeated, and the design variable b_i ($i = 1, \ldots, N$) represents the location of the ith point that separates the segments of different boundary conditions as shown in Fig. 1. The nota-

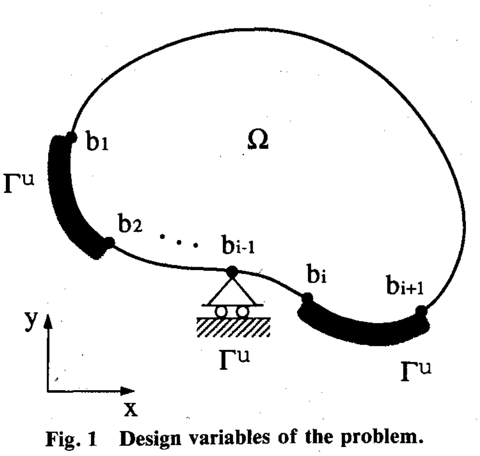


Fig. 1 Design variables of the problem.

tion I_i denotes a predetermined segment within which the point b_i will be located. Equation (1c) is the state equation of the eigenvalue problem, where U_{ad} denotes the set of admissible variations, which itself is a function of the design.

During the optimization procedure an eigenvalue crossing may arise because the patterns of mode shape are directly affected by a change of boundary conditions. In this case, it is useful to restate Eq. (1a) as a simple min (or max) problem with respect to a bound β to circumvent the inherent difficulty of nondifferentiability. To this end, the problem [Eqs. (1a–1c)] can be expressed equivalently

$$\min_{b_i} \max_{j} \phi_j$$

with constraints [Eqs. (1b) and (1c)], where $\phi_j = -\zeta_j$ or $\phi_j = \zeta_j^{-1}$. Accordingly, the optimal design problem is restated as follows,

$$\min \beta \tag{2a}$$

subject to

$$\phi_j - \beta \leq 0, \qquad j = 1, \ldots, J \tag{2b}$$

with the same constraints [Eqs. (1b) and (1c)]. Here, $\beta = b_{N+1}$ is an additional design variable physically denoting the negative of the lowest fundamental frequency when $\phi_j = -\zeta_j$ is used.

Sensitivity Analysis with Changing Boundary Conditions

The theory as developed in Ref. 2 is used to derive sensitivity formulas where the material derivative concept is utilized under variational formulation. The definition of material derivative and some results are summarized.

Consider an arbitrary shape Ω with a sufficiently smooth boundary Γ. Defining a transformation field $\mathfrak{I}(x,\tau)$, so that the point $x \in \Omega$ pass to the point $x_\tau \in \Omega_\tau$

$$x_\tau \equiv \mathfrak{I}(x,\tau) \cong x + \tau V(x) \tag{3}$$

where $V(x) \equiv \partial\mathfrak{I}(x,0)/\partial\tau$ is called the design velocity field defined at $\tau = 0$. Then, the pointwise material derivative of a state variable u (if it exists) at $x \in \Omega$ is defined as

$$\dot{u}(x) = u'(x) + \nabla u^T V(x) \tag{4}$$

where $u' = \partial u/\partial\tau$ is the time derivative while the position x is held fixed and ∇u is the gradient of u. The material derivative concept can be used to find the variation of a domain functional,

$$\Psi = \iint_{\Omega_\tau} F_\tau(u_\tau)\, d\Omega_\tau \tag{5}$$

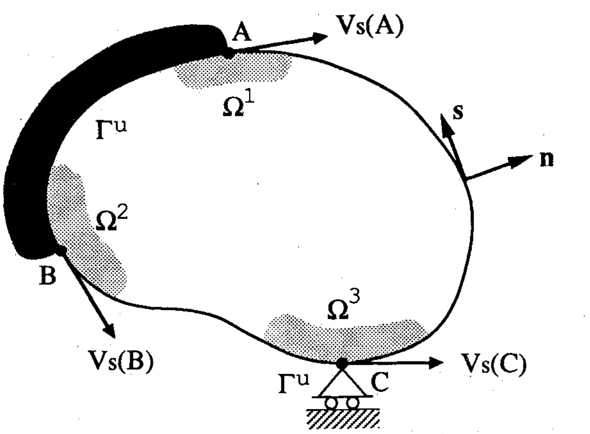


Fig. 2 Tangential velocity and changing boundary conditions.

That is,

$$\delta\Psi = \iint_\Omega \left[\frac{\partial F}{\partial u}\dot{u} + F \operatorname{div} V\right] d\Omega \tag{6a}$$

$$= \iint_\Omega \left[\frac{\partial F}{\partial u} u'\right] d\Omega + \int_\Gamma F(V^T n)\, d\Gamma \tag{6b}$$

The velocity field can be decomposed into the normal and tangential component such that

$$V = V_n n + V_s s \tag{7}$$

where V_n is the normal component of the velocity field V and V_s is tangent to the boundary. Most research in the shape design sensitivity analysis have treated the shape variation in the normal direction only. It is observed,[9,10] as shown in Fig. 2, that the tangential velocity component V_s can be used to express the variation of points A, B, and C along the boundary and hence the change of boundary conditions. In the present study, a subdomain Ω^k is defined around the point separating boundary segments of different conditions and V_s is distributed on Ω^k to describe the change of boundary conditions. The resulting sensitivity formula is expressed in a domain integral form.

Now let us derive the sensitivity formula for an eigenvalue with changing boundary conditions. The variational equation of the eigenvalue problem on Ω_τ is

$$a_\tau(u_\tau,\bar{u}_\tau) \equiv \iint_{\Omega_\tau} c_\tau(u_\tau,\bar{u}_\tau)\, d\Omega_\tau$$

$$a_\tau(u_\tau,\bar{u}_\tau) = \zeta_\tau \iint_{\Omega_\tau} e_\tau(u_\tau,\bar{u}_\tau)\, d\Omega_\tau \equiv \zeta_\tau d_\tau(u_\tau,\bar{u}_\tau) \qquad \text{for all } \bar{u}_\tau \in U_{ad}^\tau \tag{8}$$

with the normalizing condition

$$d_\tau(u_\tau,u_\tau) = 1 \tag{9}$$

where U_{ad}^τ is the space of kinematically admissible displacements for perturbed boundary condition and $c(.,.)$ and $e(.,.)$ are symmetric bilinear mappings.

The variation of the eigenvalue ζ with respect to the change of boundary conditions can be obtained by differentiating the variational eigenvalue equation and applying the same procedure as the shape sensitivity analysis in Ref. 2, but with the tangential velocity field nonzero on subdomain Ω^k around the point separating boundary segments of different conditions. Then, the eigenvalue design sensitivity formula for the changing boundary conditions is expressed as follows:

$$\zeta' = a'(u,u) - \zeta d'(u,u)$$

$$= 2\iint_{\Omega^k} [-c(u, \nabla u^T V) + \zeta e(u, \nabla u^T V)]\, d\Omega$$

$$+ \iint_{\Omega^k} \operatorname{div}[c(u,u)V - \zeta e(u,u)V]\, d\Omega \tag{10}$$

Note that the eigenvalue sensitivity formula derived earlier is valid only for a distinct eigenvalue. For m-fold multiple eigenvalues, the boundary condition sensitivity formula is the same as that derived in Ref. 2, except that now the tangential component is included. Then, the directional derivatives of multiple eigenvalues for changing boundary conditions are the eigenvalues of the matrix $\mathfrak{M}_{ij}$ such that

$$\mathfrak{M}_{ij} = a'(u^i,u^j) - \zeta d'(u^i,u^j)$$

$$= \iint_{\Omega^k} [-c(\nabla u^{i^T} V, u^j) - c(u^i, \nabla u^{j^T} V)$$

$$+ \zeta\{e(\nabla u^{i^T} V, u^j) + e(u^i, \nabla u^{j^T} V)\}]\, d\Omega$$

$$+ \iint_{\Omega^k} \operatorname{div}[c(u^i,u^j)V - \zeta e(u^i,u^j)V]\, d\Omega \qquad i,j = 1, \ldots, m \tag{11}$$

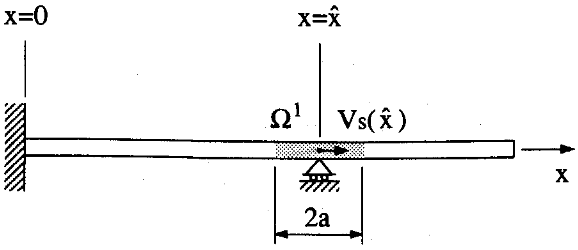


Fig. 3 Variation of boundary condition for beam.

where u^i, $i = 1, \dots, m$ are the eigenfunctions corresponding to the m-fold eigenvalue ζ.

The key point here is that a proper selection of the design velocity field, i.e., a proper distribution of the tangential velocity component V_s on the subdomain Ω^k allows one to improve the boundary conditions. As compared with those based on boundary integral equations,[9,10] the tangential components needs to be explicitly defined continuously on the domain.

Analytical Examples of Sensitivity Analysis

The general sensitivity expressions presented earlier are applied to free-vibration problems of beams and plates including shear deformation and rotary inertia effects.

Flexural Vibration of Beam

Consider the free-vibration problem of a beam with kinematic boundary conditions at $x = 0$ and $x = \hat{x}$ as shown in Fig. 3. The energy bilinear form and bilinear form due to the mass effect are[11]

$$a(u,\bar{u}) \equiv \int_0^L EI\theta_{,x}\bar{\theta}_{,x}\,\mathrm{d}x + \int_0^L GA\kappa(w_{,x}-\theta)(\bar{w}_{,x}-\bar{\theta})\,\mathrm{d}x \tag{12}$$

$$d(u,\bar{u}) \equiv \int_0^L \rho A w\bar{w}\,\mathrm{d}x + \int_0^L \rho I\theta\bar{\theta}\,\mathrm{d}x \tag{13}$$

where L, A, I, E, G, κ, and ρ are the length, cross-sectional area, moment of inertia, Young's modulus, shear modulus, shear correction factor, and material density, respectively. In Eqs. (12) and (13), w is lateral displacement, θ is the total rotation of the plane originally normal to the neutral axis of the beam, and $u = [w\theta]^T$. The subscripts following commas denote partial derivatives with respect to coordinates of the Cartesian system. The functions, $u = [w\theta]^T$ in U_{ad} must satisfy

$$w = \theta = 0 \qquad \text{at} \qquad x = 0 \tag{14}$$

$$w = 0 \qquad \text{at} \qquad x = \hat{x} \tag{15}$$

Consider the variation of eigenvalues when the simply supported location $\hat{x}$ is changed to $\hat{x}_\tau = \hat{x} + \tau V_s(\hat{x})$. The variation of boundary condition is expressed by $V_s(\hat{x})$ and subdomain Ω^1 around $\hat{x}$ is defined to distribute a regular velocity field as shown in Fig. 3. For this change of boundary condition, the first variation of the bilinear forms in Eq. (10) can be obtained

$$\begin{aligned} a'(u,u) = &-2\int_{\hat{x}-a}^{\hat{x}+a} [EI(\theta_{,x}V)_{,x}\theta_{,x} + GA\kappa\{(w_{,x}V)_{,x} \\ &- \theta_{,x}V\}(w_{,x}-\theta)]\,\mathrm{d}x + \int_{\hat{x}-a}^{\hat{x}+a} \mathrm{div}[\{EI\theta_{,x}^2 + GA\kappa(w_{,x} \\ &- \theta)^2\}V]\,\mathrm{d}x \end{aligned} \tag{16}$$

$$\begin{aligned} d'(u,u) = &-2\int_{\hat{x}-a}^{\hat{x}+a} [\rho A(w_{,x}V)w + \rho I(\theta_{,x}V)\theta]\,\mathrm{d}x \\ &+ \int_{\hat{x}-a}^{\hat{x}+a} \mathrm{div}[\{\rho Aw^2 + \rho I\theta^2\}V]\,\mathrm{d}x \end{aligned} \tag{17}$$

Therefore, the sensitivity expression for the eigenvalue with respect to the changing support location is obtained from Eq. (10) as follows:

$$\begin{aligned} \zeta' = &-\int_{\hat{x}-a}^{\hat{x}+a} [EI\theta_{,x}^2 + GA\kappa(w_{,x}^2 - \theta^2)]V_{,x}\,\mathrm{d}x \\ &- \zeta\int_{\hat{x}-a}^{\hat{x}+a} [\rho Aw^2 + \rho I\theta^2]V_{,x}\,\mathrm{d}x \end{aligned} \tag{18}$$

For m-fold multiple eigenvalues, one can use Eq. (11),

$$\begin{aligned} \mathfrak{M}_{ij} = &-\int_{\hat{x}-a}^{\hat{x}+a} [EI\theta^i_{,x}\theta^j_{,x} + GA\kappa(w^i_{,x}w^j_{,x} - \theta^i\theta^j)]V_{,x}\,\mathrm{d}x \\ &- \zeta\int_{\hat{x}-a}^{\hat{x}+a} [\rho Aw^iw^j + \rho I\theta^i\theta^j]V_{,x}\,\mathrm{d}x \qquad i,j = 1,\dots,m \end{aligned} \tag{19}$$

where the eigenvalues of the matrix $\mathfrak{M}_{ij}$ are the directional derivatives of the multiple eigenvalue with respect to the change of support location.

Flexural Vibration of Plate

Consider the free-vibration problem of a plate with kinematically constrained boundary Γ^u and free boundary $\Gamma^f(\Gamma = \Gamma^u \cup \Gamma^f)$ as shown in Fig. 4. The energy bilinear form and the bilinear form due to the mass effect are[11]

$$\begin{aligned} a(u,\bar{u}) \equiv &\iint_\Omega \hat{D}_b\bigg[(\theta_{x,x} + \nu\theta_{y,y})\bar{\theta}_{x,x} + (\nu\theta_{x,x} + \theta_{y,y})\bar{\theta}_{y,y} \\ &+ \frac{1-\nu}{2}(\theta_{x,y} + \theta_{y,x})(\bar{\theta}_{x,y} + \bar{\theta}_{y,x})\bigg]\,\mathrm{d}\Omega + \iint_\Omega \hat{D}_s[(w_{,x} \\ &- \theta_x)(\bar{w}_{,x} - \bar{\theta}_x) + (w_{,y} - \theta_{,y})(\bar{w}_{,y} - \bar{\theta}_y)]\,\mathrm{d}\Omega \end{aligned} \tag{20}$$

$$d(u,\bar{u}) \equiv \iint_\Omega \left[\rho hw\bar{w} + \frac{\rho h^3}{12}\theta_x\bar{\theta}_x + \frac{\rho h^3}{12}\theta_y\bar{\theta}_y\right]\mathrm{d}\Omega \tag{21}$$

where $\hat{D}_b = Eh^3/12(1-\nu^2)$, $\hat{D}_s = Eh\kappa/2(1+\nu)$, and ν and h are Poisson's ratio, and thickness, respectively. In Eqs. (20) and (21), w is lateral displacement, θ_x and θ_y are the rotations of the normal to the undeformed middle surface in the x-z and y-z planes, respectively, and $u = [w\theta_x\theta_y]^T$. The functions $u = [w\theta_x\theta_y]^T$ in U_{ad} must satisfy

$$w = \theta_x = \theta_y = 0 \qquad \text{on} \qquad \Gamma^u \tag{22}$$

Consider the variation of eigenvalues when the part of clamped boundary Γ^u is increased, whereas the part of free boundary Γ^f is decreased. The change of boundary condition can be represented by the tangential velocity field $V_s(\hat{X}_A)$. The subdomain Ω^1 around $\hat{X}_A$ is defined to distribute a regular

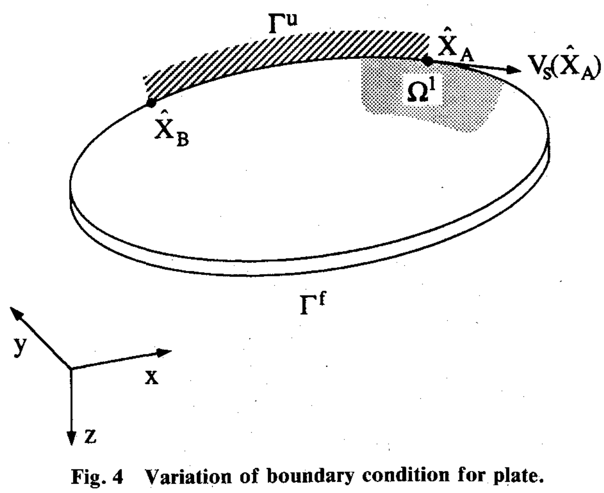


Fig. 4 Variation of boundary condition for plate.

velocity field as shown in Fig. 4. By a procedure similar to that used for the beam problem, one obtains

$$\zeta' = -2\int\int_{\Omega^1} [M_x^i\Phi_x^i + M_y^i\Phi_y^i + M_{xy}^i\Phi_{xy}^i]\, d\Omega + \Delta_B^{ii}$$

$$-2\int\int_{\Omega^1} [N_x^i\Psi_x^i + N_y^i\Psi_y^i]\, d\Omega + \Delta_S^{ii} - \zeta\Delta_I^{ii} \qquad (23)$$

where V_x and V_y are components of tangential velocity $V_s(\hat{X}_A)$ along the x and y directions. Detailed expressions of $M_x^i, \ldots, \Delta_I^{ii}$ are given in the Appendix. For m-fold multiple eigenvalues, the matrix $\mathfrak{M}_{ij}$ for sensitivity analysis can be expressed by using Eq. (11)

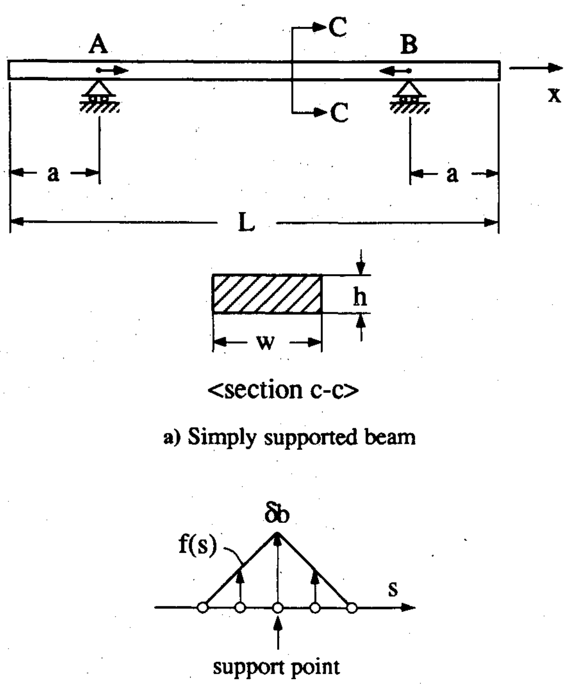
$$\mathfrak{M}_{ij} = -\int\int_{\Omega^1} [M_x^i\Phi_x^j + M_y^i\Phi_y^j + M_{xy}^i\Phi_{xy}^j]\, d\Omega$$

$$-\int\int_{\Omega^1} [M_x^j\Phi_x^i + M_y^j\Phi_y^i + M_{xy}^j\Phi_{xy}^i]\, d\Omega + \Delta_B^{ij}$$

$$-\int\int_{\Omega^1} [N_x^i\Psi_x^j + N_y^i\Psi_y^j]\, d\Omega - \int\int_{\Omega^1} [N_x^j\Psi_x^i + N_y^j\Psi_y^i]\, d\Omega$$

$$+\Delta_S^{ij} - \zeta\Delta_I^{ij} \qquad i, j = 1, \ldots, m \qquad (24)$$

where the eigenvalues of the matrix $\mathfrak{M}_{ij}$ are the directional derivatives of the multiple eigenvalues with respect to the change of a boundary condition.



b) Tangential velocity field around the support position

Fig. 5 Simply supported beam problem.

Numerical Examples

Simply Supported Beam Problem

The problem considered here is to find the best support locations of a simply supported beam as shown in Fig. 5a, which gives the maximum fundamental frequency. The dimensions are $L = 1$ m, $a = 0.2$ m, $w = 0.03$ m, and $h = 0.01$ m. Young's modulus, shear modulus, and mass density are $E = 73$ GPa, $G = 28$ GPa, and $\rho = 2800$ kg/m^3, respectively. The support points are to be varied, whereas the length of the beam is fixed. A mesh of 10 quadratic isoparametric beam elements has been take for eigenvalue analysis using the subspace iteration method.[11] The distances between support points and the ends of the beam are chosen as design variables. To describe the change of the support location in the direction of x, a velocity field is taken as nonzero between the support point and two neighboring midnodes. Theoretically the velocity field can be taken arbitrarily. It should, however, be regular enough to satisfy the requirement of the derivatives.[2] A linear distribution is used in this study, as shown in Fig. 5b. It should be noted that in this figure, the magnitude of the velocity field is drawn perpendicularly instead of tangentially.

To compare the accuracy of the sensitivity calculation with that of the finite difference, define f_i and f_m as the natural frequencies for the initial and modified boundary conditions, respectively. Let Δf be $f_m - f_i$ and f' be the values determined by the present method. Numerical results for the first five natural frequencies with a 0.1% change of support point A, i.e., $\delta b_1 = 0.001 b_1$, are shown in Table 1. The ratio $f'/\Delta f \times 100$ is used as a measure of the consistency of design sensitivity calculation with the finite difference; i.e., 100% means that the change predicted by the formula is the same as the increment of the frequency. A good agreement within 0.5% has been obtained for all natural frequencies. Figure 6 shows the fundamental natural frequency and its sensitivity with respect to support location. A refined mesh with 20 elements is employed for the analysis.

The optimal support location of the beam is obtained by application of the sensitivity formula to an iterative optimization algorithm. Move limits are imposed on each support

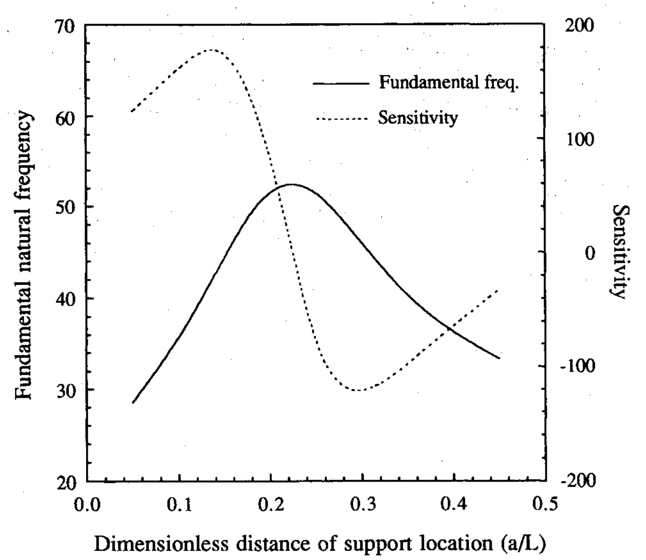


Fig. 6 Fundamental natural frequency and its sensitivity vs support location of beam.

Table 1 Sensitivity results of simply supported beam problem (Δf calculated with 0.1% design perturbation)

Mode	f_i	f_m	Δf	f'	$f'/\Delta f \times 100$
1	51.672083	51.680254	0.8171E − 02	0.8199E − 02	100.34
2	114.853215	114.790955	− 0.6226E − 01	− 0.6223E − 01	99.95
3	181.102949	181.059647	− 0.4330E − 01	− 0.4358E − 01	100.65
4	374.026002	374.137502	0.1115E + 00	0.1113E + 00	99.82
5	727.028201	727.196210	0.1680E + 00	0.1681E + 00	100.00

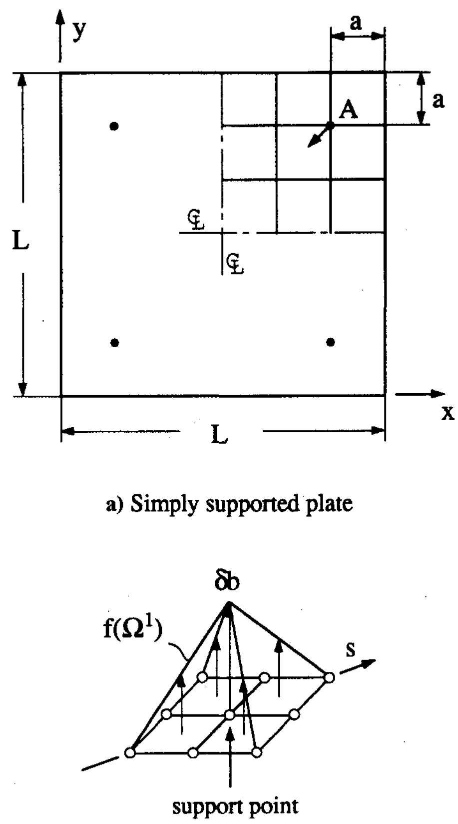
point to prevent severe distortion of finite elements. In this example, allowable move limits are taken as half of the length of a neighboring element at its support point. Support positions are restricted to be symmetric. Nonsymmetric support was also considered, and the same conclusion was reached as was reached in the symmetric case. During the optimization process, the support location, taken as a nodal point, was reselected because otherwise the changed mesh could become too distorted. The values of the fundamental natural frequency and dimensionless support locations at the initial and optimum design are listed in Table 2. It is noted that one can also read the approximate optimum design from Fig. 6. They correspond to the nodes of the beam under free-free conditions, and hence the optimum frequency is the same for this free-free beam.

Simply Supported Plate Problem

A square plate, symmetrically supported at four points on the diagonals, is considered in this example. Figure 7a shows the geometry of the plate and the finite element model of a quarter. The subspace iteration method is used. The dimensions are $L = 30.5$ cm, $a = 5.083$ cm, and the thickness is 0.328 cm. The material properties of Young's modulus, Poisson's ratio, and mass density are $E = 73.1$ GPa, $\nu = 0.3$, and $\rho = 2821$ kg/m^3, respectively. The plate is modeled by 36 uniformly discretized quadratic isoparametric elements. The simply supported points are to be varied symmetrically along the diagonal such that the fundamental natural frequency is maximized, whereas the shape of the plate is fixed. The design variables for this problem are defined as the distances between support points and the corners of the plate. The velocity field is defined as shown in Fig. 7b over a subdomain Ω^1 of four finite elements around the moving point. As before, the magnitude of the velocity field is drawn perpendicularly to the direction of s instead of tangentially.

The sensitivities of the first five natural frequencies are calculated for the change of support location A. The numerical results obtained with 0.05% change of support of point A are shown in Table 3. It should be noted that there is a pair of double eigenvalues in Table 3 due to the symmetry of the plate. If an unsymmetric variation of the boundary condition were allowed, this double eigenvalue would be split in two. Knowing the possibility of multiple eigenvalues exists, the sensitivity analysis for the third and fourth natural frequency is performed by using the sensitivity formula for a multiple eigenvalue. Very good agreement is obtained. The worst accuracy of the fifth frequency could be improved by choosing a smaller perturbation for the finite difference calculation.

The optimization problem considered here is to find the best location of four simply supported points such that the fundamental natural frequency becomes maximum. The support locations are restricted to symmetry on the diagonal. The optimization is started at two different initial designs. Results are listed in Table 4 and the first two modal patterns at the initial and the optimum design are shown in Fig. 8. From these results, it is noticeable that the patterns of mode shapes are significantly influenced by the boundary conditions. However, although the optimum support locations are much different, the two local solutions obtained give nearly the same fundamental frequency. For the purpose of illustration, the behavior of the natural frequencies for four modal patterns are investigated by moving the support positions along the diagonal with refined finite element meshes. The result is shown in Fig. 9. The interchange of mode shapes is seen at points A, B, and C for the fundamental natural frequency. The abscissas of points A and B are calculated by the optimization method by imposing the equality constraint $f_1 = f_2$. The calculated positions are $a/L = 0.169$ and $a/L = 0.295$, obtained by using different initial designs. The fundamental natural frequency is not changed with respect to the change of support locations between A and B, therefore the solution of optimization is not unique. Rather, the interval of locations between A and B are all solutions. As in the previous example,



a) Simply supported plate

b) Tangential velocity field around the support position

Fig. 7 Simply supported square plate problem.

Table 2 Optimum support location and fundamental natural frequency for simply supported beam

	Initial design	Optimum design
a/L	0.100	0.224
f_1, Hz	35.890	52.710

Table 3 Sensitivity results of simply supported square plate problem (Δf calculated with 0.05% design perturbation)

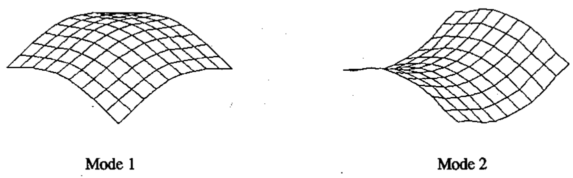
Mode	f_i	f_m	Δf	f'	$f'/\Delta f \times 100$
1	167.816173	167.837135	$0.2096E-01$	$0.2097E-01$	100.05
2	169.451447	169.451452	$0.5338E-05$	$0.5298E-05$	99.25
3	255.438289[a]	255.438290	$0.1792E-05$	$0.1738E-05$	96.99
4	255.438289[b]	255.469235	$0.3095E-01$	$0.3094E-01$	99.97
5	502.492472	502.493296	$0.8241E-03$	$0.1274E-02$	154.59
		(502.492709	$0.2372E-03$	$0.2550E-03$	107.50)[c]

[a,b]Double eigenvalue.
[c]Values obtained with 0.01% change for the fifth natural frequency.

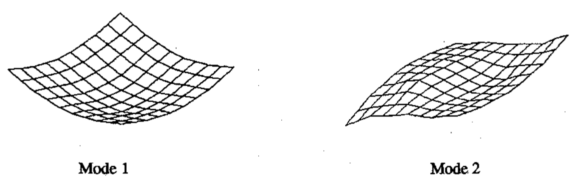
it is noted that the optimum frequency is the same for that of a completely free square plate that has the nodal lines running along the diagonals. Comparing the results obtained in Refs. 6 and 7 and here, we can note that all of the optimal support locations are between A and B. In the present approach, the situation of eigenvalue crossing is identified and well-treated, which has not been possible in the literature reported. In addition, the proposed method can be easily extended to optimize higher frequencies.

Welded Plate Problem

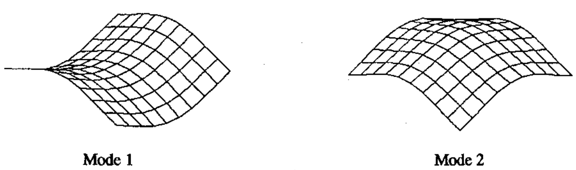
A plate that is to be welded to a rigid structure is shown in Fig. 10a. An extension of the welded length means an increase of the clamped boundary Γ^u and a decrease of the free boundary Γ^f, that is, a change of the nature of the boundary from Γ^f to Γ^u. It can be represented with a tangential velocity around the tip of the welded zone. The dimensions are $L_x = 40.6$ cm, $L_y = 30.5$ cm, $a = 10.2$ cm, and the thickness is 0.328 cm. The plate is modeled with 48 quadratic isoparametric elements. The material properties are identical to those of previous examples. A nonzero tangential velocity field is assigned to a subdomains in the present example: a two-element zone around the tip of the welded zone as shown in Fig. 10b. As in the previous examples the magnitude of the velocity field is drawn perpendicularly to the direction of s instead of tangentially. The length of the welded zone is taken as a design variable and the variations of the first five natural frequencies with a 0.1% change in welded length are predicted by the present method. Table 5 documents the numerical results f'



a) First two modal patterns at initial design (a/L=0.166)



b) First two modal patterns at initial design (a/L=0.333)



c) First two modal patterns at optimum design (a/L=0.169, a/L=0.274)

Fig. 8 Modal patterns at initial and optimal design.

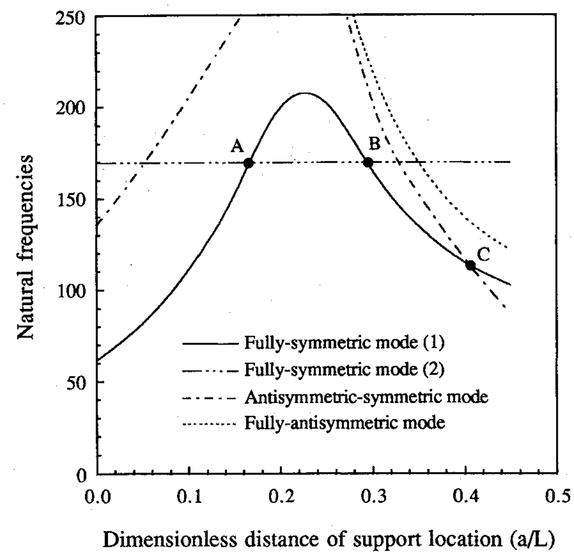
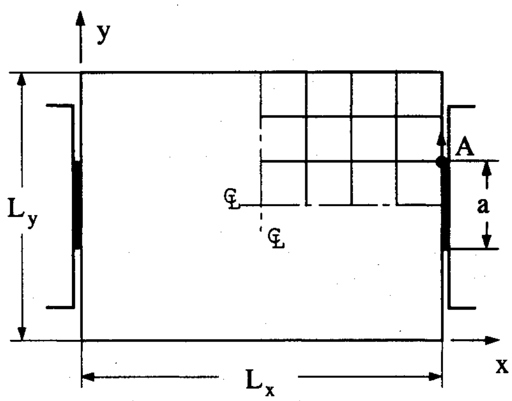


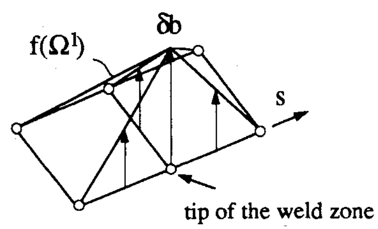
Fig. 9 Fundamental natural frequency vs support location of plate.

Table 4 Optimum support locations and fundamental natural frequencies for simply supported square plate

	Initial design	Optimum design
a/L	0.166	0.169
f_1, Hz	167.810	169.460
f/L	0.333	0.274
f_1, Hz	144.790	169.670



a) Welded plate



b) Tangential velocity field around the tip zone

Fig. 10 Partially welded rectangular plate problem.

Table 5 Sensitivity results of welded rectangular plate problem (Δf calculated with 0.1% design perturbation)

Mode	f_i	f_m	Δf	f'	$f'/\Delta f \times 100$
1	83.030146	83.039674	$0.9528E-02$	$0.9552E-02$	100.25
2	90.049283	90.067408	$0.1812E-01$	$0.1811E-01$	99.93
3	203.113301	203.150839	$0.3754E-01$	$0.3799E-01$	101.20
4	207.421849	207.466583	$0.4473E-01$	$0.4443E-01$	99.31
5	221.162976	221.192826	$0.2985E-01$	$0.2970E-01$	99.50

predicted from the sensitivity formula and those Δf predicted by the finite differencing. Fairly accurate results are obtained. It has been numerically asserted that the size of the subdomain taken for the nonzero design velocity has no significant influence on the final result.

Conclusions

An optimal design problem has been formulated for vibrating structures by taking the boundary conditions as design variables. The eigenvalue sensitivity formula with changing boundary conditions is derived in variational form by using the material derivative concept. The change of boundary conditions is described using the tangential component of a velocity field continuously distributed over a subdomain. To illustrate the method, explicit sensitivity formulas of eigenvalues are derived for beam and plate problems considering shear deformation and rotary inertia effects. Numerical results are presented for simply supported and clamped boundary conditions, including the cases with repeated eigenvalues. The sensitivity calculation is shown to be accurate when compared with that obtained by finite difference. Optimum boundary conditions are obtained for a simply supported beam and a plate by maximizing the fundamental natural frequency. It is shown that the new boundary condition optimization problem proposed here can be solved by using the shape sensitivity analysis with the tangential component of design velocity.

Appendix

The notations from M_x^i to Δ_I^{ij} defined in Eqs. (23) and (24) are shown next. The superscripts denote that the values are calculated for eigenfunctions corresponding to the ith or jth eigenvalue.

$$M_x^i = \hat{D}_b(\theta^i_{x,x} + \nu\theta^i_{y,y}) \tag{A1}$$

$$M_y^i = \hat{D}_b(\nu\theta^i_{x,x} + \theta^i_{y,y}) \tag{A2}$$

$$M_{xy}^i = \frac{\hat{D}_b(1-\nu)}{2}(\theta^i_{x,y} + \theta^i_{y,x}) \tag{A3}$$

$$\Phi_x^i = (\theta^i_{x,x}V_{x,x} + \theta^i_{x,y}V_{y,x}) \tag{A4}$$

$$\Phi_y^i = (\theta^i_{y,x}V_{x,y} + \theta^i_{y,y}V_{y,y}) \tag{A5}$$

$$\Phi_{xy}^i = (\theta^i_{x,x}V_{x,y} + \theta^i_{x,y}V_{y,y}) + (\theta^i_{y,x}V_{x,x} + \theta^i_{y,y}V_{y,x}) \tag{A6}$$

$$N_x^i = \hat{D}_s(w^i_{,x} - \theta^i_x) \tag{A7}$$

$$N_y^i = \hat{D}_s(w^i_{,y} - \theta^i_y) \tag{A8}$$

$$\Psi_x^i = (w^i_{,x}V_{x,x} + w^i_{,y}V_{y,x}) \tag{A9}$$

$$\Psi_y^i = (w^i_{,x}V_{x,y} + w^i_{,y}V_{y,y}) \tag{A10}$$

$$\Delta_B^{ij} = \iint_{\Omega^1}\left[\left\{M_x^i\theta^j_{x,x} + M_y^i\theta^j_{y,y} + M_{xy}^i\left(\theta^j_{x,y} + \theta^j_{y,x}\right)\right\}(\operatorname{div}V)\right] d\Omega \tag{A11}$$

$$\Delta_S^{ij} = \iint_{\Omega^1}\left[\left\{N_x^i\left(w^j_{,x} - \theta^j_x\right) + N_y^i\left(w^j_{,y} - \theta^j_y\right)\right\}(\operatorname{div}V)\right] d\Omega \tag{A12}$$

$$\Delta_I^{ij} = \iint_{\Omega^1}\left(\rho h w^i w^j + \frac{\rho h^3}{12}\theta^i_x\theta^j_x + \frac{\rho h^3}{12}\theta^i_y\theta^j_y\right)(\operatorname{div}V)\, d\Omega \tag{A13}$$

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